Power Law Fluids with Random Forcing

Nobuo Yoshida[1]

This is a joint work with Yutaka Terasawa (Tohoku Univ).
We consider viscous, imcompressible fluid subject to a random perturbation. As a part of idealization, the container of the fluid is supposed to be the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1]^d$. We assume that the extra stress tensor

$$\tau(v) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$$

depends on the velocity field $v : \mathbb{T}^d \to \mathbb{R}^d$ via the following power law: for $\nu > 0$ (the kinematic viscosity), $p > 1$ (the power index) and a constant $\kappa > 0$,

$$\tau(v) = 2\nu(\kappa + |e(v)|^2)^{\frac{p-2}{2}} e(v) \text{ with } e(v) = \left(\frac{\partial_i v_j + \partial_j v_i}{2}\right). \quad (0.1)$$

The linearly dependent case $p = 2$ is the *Newtonian fluid*, which is described by the Navier-Stokes equation, the special case of (0.2)–(0.3) below. On the other hand, both the *shear thinning* ($p < 2$) and the *shear thickenning* ($p > 2$) cases are considered in many fields in science and engneering (e.g., suspension of cement in water, paper pulp in water, latex paint, blood flow, earth's mantle convection, cf. [MNRR96, Wi09]).

Given an initial velocity $u_0 : \mathbb{T}^d \to \mathbb{R}^d$, the dynamics of the fluid is described by the following SPDE:

$$\text{div } u = 0, \quad (0.2)$$

$$\partial_t u_i + \sum_{j=1}^d u_j \partial_j u_i = -\partial_i \Pi + \sum_{j=1}^d \partial_j \tau_{ij}(u) + \partial_t W_i, \quad i = 1, ..., d. \quad (0.3)$$

The unknown process in the SPDE are the velocity field $u = u(t,x) = (u_i(t,x))_{i=1}^d$ and the pressure $\Pi = \Pi(t,x)$. The Brownian motion $W = W(t,x) = (W_i(t,x))_{i=1}^d$ with values in $L^2(\mathbb{T}^d \to \mathbb{R}^d)$ is added as the random perturbation. Physical interpretation of (0.2) and (0.3) are the conservation laws of the mass and the momentum, repectively. Note that the SPDE (0.2)–(0.3) for the case $p = 2$ is the stochastic Navier-Stokes equation.

For certain ranges of the power p (e.g.,$p \in (3/2, \infty)$ for $d = 2$, $p \in (9/5, 6)$ for $d = 3$), we construct a weak solution to the SPDE (0.2)–(0.3) globally in time. For $d = 2, 3$, this generalizes the known result for the stochastic Navier-Stokes equation ([Fl08] and references therein). Also, by considering the degenerate noise, our result recovers the PDE result for $p \neq 2$ [MNRR96].

References

[Fl08] Flandoli, Franco : An introduction to 3D stochastic fluid dynamics. SPDE in hydrodynamic: recent progress and prospects, 51–150, Lecture Notes in Math., 1942, Springer, Berlin, 2008.

[MNRR96] Málek, J.; Nečas, J.; Rokyta, M.; Růžička, M.: Weak and measure-valued solutions to evolutionary PDEs. Applied Mathematics and Mathematical Computation, 13. Chapman & Hall, London, 1996. xii+317 pp. ISBN: 0-412-57750-X

[Wi09] Non-Newtonian fluid–Wikipedia.

[1]Division of Mathematics Graduate School of Science Kyoto University, Kyoto 606-8502, Japan. email: nobuo@math.kyoto-u.ac.jp URL: http://www.math.kyoto-u.ac.jp/~nobuo/