2D Ising percolation near critical external fields

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This talk is based on the joint work with Yasunari Higuchi (Kobe University) and Yu Zhang (University of Colorado).

We consider the percolation problem for Ising model on the two-dimensional square lattice $\mathbf{Z}^2$. For $T > T_c$ and $h \in \mathbf{R}$, there exists a unique Gibbs measure $\mu_{T,h}$. The (+)-cluster containing the origin is denoted by $\mathbf{C}_0^+$. For each $T > 0$, the critical external field is defined by

$$h_c(T) := \inf\{h : \mu_{T,h}(\#\mathbf{C}_0^+ = \infty) > 0\}.$$

It is known (see e.g. [1]) that $h_c(T) > 0$ whenever $T > T_c$. Hereafter we fix a $T > T_c$, and abbreviate $\mu_{T,h}$ to μ_h and $h_c(T)$ to h_c, respectively. The expectation under μ_h is denoted by E_h.

The following power laws are widely believed to hold:

- Percolation probability:

$$\theta(h) := \mu_h(\#\mathbf{C}_0^+ = \infty) \approx (h - h_c)^{\beta} \quad \text{as } h \searrow h_c.$$

- Mean cluster size:

$$\chi(h) := E_h[\#\mathbf{C}_0^+ : \#\mathbf{C}_0^+ < \infty] \approx |h - h_c|^{-\gamma} \quad \text{as } h \to h_c.$$

- Correlation length:

$$\xi(h) := \left[\frac{1}{\chi(h)} \sum_{v \in \mathbf{Z}^2} |v|^2 \mu_h\big(\mathbf{O} \overset{+}{\leftrightarrow} v,\, \#\mathbf{C}_0^+ < \infty\big)\right]^{1/2} \approx |h - h_c|^{-\nu} \quad \text{as } h \to h_c.$$

  * For $S(n) = [-n, n]^2$, we define

$$L(h, \varepsilon_0) := \begin{cases} \min\big\{n : \mu_h\big(\text{LS of } S(n) \overset{+}{\leftrightarrow} \text{RS of } S(n)\big) \geq 1 - \varepsilon_0\big\} & (h > h_c), \\ \min\big\{n : \mu_h\big(\text{LS of } S(n) \overset{+}{\leftrightarrow} \text{RS of } S(n)\big) \leq \varepsilon_0\big\} & (h < h_c). \end{cases}$$

  Then $\xi(h) \asymp L(h, \varepsilon_0)$.

- One-arm probability: $\pi_{h_c}(n) := \mu_{h_c}\big(\mathbf{O} \overset{+}{\leftrightarrow} \partial S(n)\big) \approx n^{-1/\delta_{\mathrm{r}}}$.

- Connectivity function: $\tau_{h_c}(n) := \mu_{h_c}\{\mathbf{O} \overset{+}{\leftrightarrow} (n, 0)\} \approx n^{-\eta}$.

We extend some of Kesten's scaling relations [2] for 2D Bernoulli percolation to our case. Our main result is as follows.
Theorem. 1) If

$$\pi_{h_c}(n) \approx n^{-1/\delta_{\mathrm{r}}} \qquad \text{or} \qquad \tau_{h_c}(n) \approx n^{-\eta} \tag{1}$$

holds, then both statements as well as

$$\mu_{h_c}\{\#\mathbf{C}_0^+ \geq n\} \approx n^{-1/\delta}$$

hold. Moreover,

$$\theta(h) \asymp L(h,\varepsilon_0)^{-1/\delta_{\mathrm{r}}} = L(h,\varepsilon_0)^{-2/(\delta+1)}$$

and

$$\eta = \frac{2}{\delta_{\mathrm{r}}}, \qquad \delta = 2\delta_{\mathrm{r}} - 1 = \frac{4}{\eta} - 1.$$

In addition, if

$$\xi(h) \approx |h - h_c|^{-\nu}, \tag{2}$$

then $\beta = \dfrac{2\nu}{\delta+1}$.

2) Assume that (1) and (2) hold.

- For $t \geq 2$, $\dfrac{E_h[(\#\mathbf{C}_0^+)^t : \#\mathbf{C}_0^+ < \infty]}{E_h[(\#\mathbf{C}_0^+)^{t-1} : \#\mathbf{C}_0^+ < \infty]} \approx \xi(h)^2 \pi_{h_c}(\xi(h))$,
- For $t > 0$, $\left[\dfrac{1}{\chi(h)} \displaystyle\sum_{v \in \mathbf{Z}^2} |v|^t \mu_h\big(\mathbf{O} \overset{+}{\leftrightarrow} v,\, \#\mathbf{C}_0^+ < \infty\big)\right]^{1/t} \asymp \xi(h)$.

Moreover,

- For $k \geq 2$, $\dfrac{E_h[(\#\mathbf{C}_0^+)^k : \#\mathbf{C}_0^+ < \infty]}{E_h[(\#\mathbf{C}_0^+)^{k-1} : \#\mathbf{C}_0^+ < \infty]} \approx |h - h_c|^{-\Delta_k}$,
- For $k \geq 1$, $\left[\dfrac{1}{\chi(h)} \displaystyle\sum_{v \in \mathbf{Z}^2} |v|^k \mu_h\big(\mathbf{O} \overset{+}{\leftrightarrow} v,\, \#\mathbf{C}_0^+ < \infty\big)\right]^{1/k} \approx |h - h_c|^{-\nu_k}$,

and

$$\gamma = 2\nu\frac{\delta-1}{\delta+1}, \qquad \Delta_k = 2\nu\frac{\delta}{\delta+1} \quad (k \geq 2), \qquad \nu_k = \nu \quad (k \geq 1).$$

References

[1] Higuchi, Y. : *Sugaku Expositions* **10**, 143–158, (1997).
[2] Kesten, H. : *Probab. Theory Related Fields* **73**, 369–394, (1986); *IMA Vol. Math. Appl.* **8**, 203–212, (1987); *Comm. Math. Phys.* **109**, 109–156, (1987).