Hydrodynamic limit and fluctuations for an evolutional model of two-dimensional Young diagrams

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We construct dynamics of two-dimensional Young diagrams, which are naturally associated with their grandcanonical ensembles and show that, as the averaged size of the diagrams diverges, the corresponding height variable converges to a solution of a certain non-linear partial differential equation under a proper hydrodynamic scaling. We discuss both uniform and restricted uniform statistics for the Young diagrams. Also, we study the corresponding dynamic fluctuation problem in a non-equilibrium situation.

To each partition $p = \{p_1 \geq p_2 \geq \cdots \geq p_j \geq 1\}$ of a positive integer n by positive integers $\{p_i\}_{i=1}^j$ (i.e., $n = \sum_{i=1}^j p_i$), the height function of the Young diagram is defined by $\psi_p(u) = \sum_{i=1}^j 1_{\{u<p_i\}}$ for $u \geq 0$. For each fixed n, the uniform statistics (U-statistics in short) μ_U^n assigns an equal probability to each of possible partitions p of n, i.e., to the Young diagrams of area n. The restricted uniform statistics (RU-statistics in short) μ_R^n also assigns an equal probability, but restricting to the partitions satisfying $q = \{q_1 > q_2 > \cdots > q_j \geq 1\}$. Grandcanonical ensembles μ_U^ε and μ_R^ε with parameter $0 < \varepsilon < 1$ are defined by superposing the canonical ensembles. Vershik [1] proved that, under the canonical U- and RU- statistics $\mu_U^{N^2}$ and $\mu_R^{N^2}$ (with $n = N^2$), the law of large numbers holds as $N \to \infty$ for the scaled height variable

$$\tilde{\psi}_p^N(u) := \frac{1}{N}\psi_p(Nu), \quad u \geq 0, \tag{1}$$

and for $\tilde{\psi}_q^N(u)$ defined similarly, and the limit shapes ψ_U and ψ_R are given by

$$\psi_U(u) = -\frac{1}{\alpha}\log\left(1 - e^{-\alpha u}\right) \quad \text{and} \quad \psi_R(u) = \frac{1}{\beta}\log\left(1 + e^{-\beta u}\right), \quad u \geq 0, \tag{2}$$

with $\alpha = \pi/\sqrt{6}$ and $\beta = \pi/\sqrt{12}$, respectively. These results can be extended to the corresponding grandcanonical ensembles μ_U^ε and μ_R^ε, if the averaged size of the diagrams is N^2 under these measures.

The purpose of our talk is to study and extend these results from a dynamical point of view. First, we construct dynamics of two-dimensional Young diagrams, which have μ_U^ε and μ_R^ε as their invariant measures, respectively. Let $p_t \equiv p_t^\varepsilon = (p_i(t))_{i\in\mathbb{N}}$ be the Markov process defined by means of the infinitesimal generator $L_{\varepsilon,U}$:

$$L_{\varepsilon,U} f(p) = \sum_{i\in\mathbb{N}} \left[\varepsilon 1_{\{p_{i-1}>p_i\}}\{f(p^{i,+}) - f(p)\} + 1_{\{p_i>p_{i+1}\}}\{f(p^{i,-}) - f(p)\}\right], \tag{3}$$

where

$$p_j^{i,\pm} = \begin{cases} p_j & \text{if } j \neq i, \\ p_i \pm 1 & \text{if } j = i. \end{cases} \tag{4}$$

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In (3), we regard $p_0 = \infty$. Let $q_t \equiv q_t^\varepsilon = (q_i(t))_{i\in\mathbb{N}}$ be the Markov process with the infinitesimal generator $L_{\varepsilon,R}$:

$$(5)\quad L_{\varepsilon,R}f(q) = \sum_{i\in\mathbb{N}} \left[\varepsilon 1_{\{q_{i-1}>q_i+1\}}\{f(q^{i,+}) - f(q)\} + 1_{\{q_i>q_{i+1}+1 \text{ or } q_i=1\}}\{f(q^{i,-}) - f(q)\}\right],$$

where $q^{i,\pm}$ are defined by the formula (4) and we regard $q_0 = \infty$.

Under the diffusive scaling in space and time and choosing the parameter $\varepsilon = \varepsilon(N)$ of the grandcanonical ensembles such that the averaged size of the Young diagrams is N^2, we derive the hydrodynamic equations in the limit and show that the Vershik curves defined by (2) are actually stationary solutions to the limiting non-linear partial differential equations in both cases. Denote by $\tilde{\psi}_p^N(t,u) := \frac{1}{N}\psi_{p_{tN^2}}(Nu)$ the scaled height variable and define $\tilde{\psi}_q^N(t,u)$ similarly.

Theorem 1. *(U-case) If $\tilde{\psi}_p^N(0,u)$ converges to a function $\psi_0(u)$ in probability as $N\to\infty$, then $\tilde{\psi}_p^N(t,u)$ converges to $\psi_U(t,u)$ in probability where the limit $\psi_U(t,u)$ is the solution of the non-linear partial differential equation (PDE):*

$$(6)\qquad \begin{cases} \partial_t\psi = \{\psi'/(1-\psi')\}' + \alpha\psi'/(1-\psi'), \quad u>0, \\ \psi(0,\cdot) = \psi_0(\cdot), \\ \psi(t,0+) = \infty,\ \psi(t,\infty) = 0, \end{cases}$$

where $\partial_t\psi = \partial\psi/\partial t, \psi' = \partial\psi/\partial u (<0)$.

Theorem 2. *(RU-case) If $\tilde{\psi}_q^N(0,u)$ converges to a function $\psi_0(u)$ in probability as $N\to\infty$, then $\tilde{\psi}_q^N(t,u)$ converges to $\psi_R(t,u)$ in probability where the limit $\psi_R(t,u)$ is the solution of the non-linear partial differential equation (PDE):*

$$(7)\qquad \begin{cases} \partial_t\psi = \psi'' + \beta\psi'(1+\psi'), \quad u>0, \\ \psi(0,\cdot) = \psi_0(\cdot), \\ \psi'(t,0+) = -\dfrac{1}{2},\ \psi(t,\infty) = 0. \end{cases}$$

Next, we consider the fluctuations of $\tilde{\psi}_p^N(t,u)$ and $\tilde{\psi}_q^N(t,u)$ around their limits, respectively:

$$\Psi_p^N(t,u) := \sqrt{N}\big(\tilde{\psi}_p^N(t,u) - \psi_U(t,u)\big), \qquad \Psi_q^N(t,u) := \sqrt{N}\big(\tilde{\psi}_q^N(t,u) - \psi_R(t,u)\big).$$

By formal computations, we conjecture that $\Psi_p^N(t,u)$ and $\Psi_q^N(t,u)$ weakly converge to the solutions of the following stochastic partial differential equations respectively:

$$d\Psi(t,u) = \left(\Big(\frac{\Psi'(t,u)}{(1+\rho_U(t,u))^2}\Big)' + \alpha\frac{\Psi'(t,u)}{(1+\rho_U(t,u))^2} \right) dt + \sqrt{\frac{2\rho_U(t,u)}{1+\rho_U(t,u)}} dW(t,u)$$

for $\Psi_p^N(t,u)$ and

$$d\Psi(t,u) = \big(\Psi''(t,u) + \beta(1-2\rho_R(t,u))\Psi'(t,u)\big)\,dt + \sqrt{2\rho_R(t,u)(1-\rho_R(t,u))}dW(t,u),$$

for $\Psi_q^N(t,u)$ where $\rho_U(t,u) = -\partial_u\psi_U(t,u)$, $\rho_R(t,u) = -\partial_u\psi_R(t,u)$ and $\dot{W}(t,u)$ is the space-time white noise on $[0,T]\times\mathbb{R}_+$.

References

[1] A. Vershik, *Statistical mechanics of combinatorial partitions and their limit shapes*, Func. Anal. Appl., **30** (1996), 90–105.