Asymptotic behavior of the gyration radius for long-range self-avoiding walk and long-range oriented percolation

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Let $\alpha > 0$ and suppose that the 1-step distribution D for random walk on $\mathbb{Z}^d$ decays as $D(x) \approx |x|^{-d-\alpha}$ such that its Fourier transform $\hat{D}(k) \equiv \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x} D(x)$ satisfies

$$1 - \hat{D}(k) = v_\alpha |k|^{\alpha\wedge 2} \times \begin{cases} 1 + O(|k|^\epsilon) & (\alpha \neq 2), \\ \log \frac{1}{|k|} + O(1) & (\alpha = 2), \end{cases} \tag{1}$$

for some $v_\alpha \in (0,\infty)$ and $\epsilon > 0$. The following long-range Kac potential, for any $L \in [1,\infty)$, satisfies the above property with $v_\alpha = O(L^{\alpha\wedge 2})$ [3]:

$$D(x) = \frac{h(y/L)}{\sum_{y\in\mathbb{Z}^d} h(y/L)} \qquad (x \in \mathbb{Z}^d), \tag{2}$$

where $h(x) \equiv |x|^{-d-\alpha}(1 + O(|x|^\epsilon))$ is a rotation-invariant function on $\mathbb{R}^d$.

Let $\varphi_t(x)$ denote the two-point functions for random walk and self-avoiding walk whose 1-step distribution is given by the above D and for oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ whose bond-occupation probability for each bond $((u,s),(v,s+1))$ is given by $pD(v-u)$, independently of $s \in \mathbb{Z}_+$, where $p \geq 0$ is the percolation parameter. More precisely,

$$\varphi_t(x) = \begin{cases} \displaystyle\sum_{\substack{\omega:o\to x\\ |\omega|=t}} \prod_{s=1}^t D(\omega_s - \omega_{s-1}) & \text{(RW)}, \\ \displaystyle\sum_{\substack{\omega:o\to x\\ |\omega|=t}} \prod_{s=1}^t D(\omega_s - \omega_{s-1}) \prod_{0\leq i<j\leq t} (1-\delta_{\omega_i,\omega_j}) & \text{(SAW)}, \\ \mathbb{P}_p\big((o,0) \longrightarrow (x,t)\big) & \text{(OP)}, \end{cases} \tag{3}$$

where $\prod_{0\leq i<j\leq t}(1-\delta_{\omega_i,\omega_j})$ is the self-avoiding constraint on ω, and $\{(o,0)\to(x,t)\}$ is the event that either $(x,t) = (o,0)$ or there is a consecutive sequence of occupied bonds from $(o,0)$ to (x,t) in the time-increasing direction. The order-r gyration radius $\xi_t^{(r)}$, defined as

$$\xi_t^{(r)} = \left(\frac{\sum_{x\in\mathbb{Z}^d} |x|^r \varphi_t(x)}{\sum_{x\in\mathbb{Z}^d} \varphi_t(x)}\right)^{1/r}, \tag{4}$$

represents a typical end-to-end distance of a linear structure of length t or a typical spatial size of a cluster at time t. It has been expected (and is certainly true for random walk in any dimension) that, above the common upper-critical dimension $d_c = 2(\alpha\wedge 2)$ for self-avoiding walk and oriented percolation, for every $r \in (0,\alpha)$,

$$\xi_t^{(r)} = \begin{cases} O(t^{\frac{1}{\alpha\wedge 2}}) & (\alpha \neq 2), \\ O(\sqrt{t\log t}) & (\alpha = 2). \end{cases} \tag{5}$$

The conjecture was proved to be affirmative for self-avoiding walk, but only for small $r < \alpha\wedge 2$ [4].

In my recent joint work with L.-C. Chen [3], we have proved the following sharp asymptotics:

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Theorem 1 ([3]). *Consider the aforementioned three long-range models. For random walk in any dimension with any L, and for self-avoiding walk and critical/subcritical oriented percolation for $d > 2(\alpha \wedge 2)$ with $L \gg 1$, the following holds for every $r \in (0, \alpha)$: there are constants $C_1, C_2 = 1 + O(L^{-d})$ ($C_1 = C_2 = 1$ for random walk) and $\epsilon > 0$ such that, as $m \nearrow m_c$,*

$$\sum_{t=0}^{\infty} \sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x)\, m^n = \frac{2 \sin \frac{r\pi}{\alpha \vee 2}}{(\alpha \wedge 2) \sin \frac{r\pi}{\alpha}} \Gamma(r+1) \frac{C_1 (C_2 v_\alpha)^{\frac{r}{\alpha \wedge 2}}}{(1 - \frac{m}{m_c})^{1 + \frac{r}{\alpha \wedge 2}}} \times \begin{cases} 1 + O\big((1 - \frac{m}{m_c})^\epsilon\big) & (\alpha \neq 2), \\ \Big(\log \frac{1}{\sqrt{1 - \frac{m}{m_c}}}\Big)^{r/2} + O(1) & (\alpha = 2). \end{cases} \tag{6}$$

where m_c is the radius of convergence for the sequence $\sum_{x \in \mathbb{Z}^d} \varphi_t(x)$.

In fact, the above C_1, C_2 are the following model-dependent constants [1, 2, 4]:

$$\sum_{x \in \mathbb{Z}^d} \varphi_t(x) \underset{t \uparrow \infty}{\sim} C_1 m_c^{-t}, \qquad \frac{\sum_{x \in \mathbb{Z}^d} e^{ik_t \cdot x} \varphi_t(x)}{\sum_{x \in \mathbb{Z}^d} \varphi_t(x)} \underset{t \uparrow \infty}{\sim} e^{-C_2 |k|^{\alpha \wedge 2}}. \tag{7}$$

Theorem 2 ([3]). *Under the same condition as in Theorem 1,*

$$\frac{\sum_{x \in \mathbb{Z}^d} |x_1|^r \varphi_t(x)}{\sum_{x \in \mathbb{Z}^d} \varphi_t(x)} \underset{t \uparrow \infty}{\sim} \frac{2 \sin \frac{r\pi}{\alpha \vee 2}}{(\alpha \wedge 2) \sin \frac{r\pi}{\alpha}} \frac{\Gamma(r+1)}{\Gamma(\frac{r}{\alpha \wedge 2} + 1)} (C_2 v_\alpha)^{\frac{r}{\alpha \wedge 2}} \times \begin{cases} t^{\frac{r}{\alpha \wedge 2}} & (\alpha \neq 2), \\ (t \log \sqrt{t})^{r/2} & (\alpha = 2). \end{cases} \tag{8}$$

As far as we notice, even for random walk, the sharp asymptotics in the above two theorems are new. By $|x_1|^r \le |x|^r \le d^{r/2} \sum_{j=1}^d |x_j|^r$ and the $\mathbb{Z}^d$-symmetry of the models, Theorem 2 immediately proves the conjecture (5) for all $r \in (0, \alpha)$.

The proof is based on the derivatives of the lace expansion and the new fractional-moment analysis for the derivatives of the expansion coefficients, initiated in [2]. It is worth emphasizing that the same proof applies to finite-range models, for which α is considered to be infinity.

In the talk, I explain the general framework to treat all three models simultaneously and show some complex analysis for the derivation of the right constants in the asymptotics.

References

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