RECURRENCE THEOREM AND ERGODICITY OF QUANTUM DYNAMICS

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It is quite well-known that Hamiltonian flows in classical mechanics confined to a finite volume in phase space will eventually return to its initial state, namely, denoting the flow by T_t, we have $\liminf_{t\to\infty} d(T_t(x), x) = 0$ for almost every x. This is a typical consequence of the Poincaré recurrence theorem, and the celebrated Liouville theorem which asserts that every Hamiltonian flow preserves the Lebesgue measure. We note here that a linear motion of constant speed clearly never returns to the initial state, which means that the confining property is essential in this case.

It seems natural to try to extend this classical result to dynamics in quantum mechanics. To our knowledge, Bocchieri and Loinger [BL57] are the first peoples in this direction: Let $\Psi(t)$ be a wave function evolving in time t under a Hamiltonian H which has only discrete spectra. Then for each $\varepsilon > 0$ there exists a $T > 1$ such that $\|\Psi(T) - \Psi(0)\| < \varepsilon$. The assumption that the Hamiltonian has only discrete spectra seems natural; it is a reflection of the confining property of the classical case. It is also easy to prove that a Schrödinger flow generated by a Hamiltonian with an absolutely continuous spectrum never returns to its initial state by virtue of the Riemann–Lebesgue theorem (due to Sasaki[Sas09]).

We point out here that there is another approach to the quantum recurrence theorem; see e.g. [Duv02]. Their aim is to formulate the quantum recurrence theorem similarly to the classical one. They formulated the theorem in terms of C^*-algebra and obtained a quantum analogue of the Liouville theorem.

We will here give yet another approach which is based on a standard measure theory (elementary probability theory). It will be a simple and entirely direct extension of classical settings and it also enables us to formulate an ergodic property of the quantum dynamics. To do so we in addition need a slight stronger assumption of the spectra of the Hamiltonian and a little infinite dimensional analysis (stochastic analysis in a narrow sense).

Finally we note that Hida has studied the ergodicity of unitary (especially shift) operators on L^2-space after extending it to an operator on $\mathscr{S}'$[Hid02]. Our situation is different from his.

References

[BL57] P. Bocchieri and A. Loinger, *Quantum recurrence theorem*, Phys. Rev. **107** (1957), no. 2, 337–338.

[Duv02] Rocco Duvenhage, *Recurrence in quantum mechanics*, Intern. J. Theor. Phys **41** (2002), no. 1, 45–61.

[Hid02] Takeyuki Hida, *White noise and functional analysis*, vol. 60, SEMINAR on PROBABILITY, 2002, (in Japanese).

[Sas09] Itaru Sasaki, *A remark on Poincaré's recurrence theorem in quantum systems*, Private memorandom, 2009.