TAGGED PARTICLES OF INTERACTING BROWNIAN MOTIONS WITH THE 2D COULOMB POTENTIAL AND THE STOCHASTIC DOMINATION OF THE GINIBRE RANDOM POINT FIELD

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We prove the tagged particles of the interacting Brownian motions in $\mathbb{R}^2$ with the 2D Coulomb potential $\Psi(x) = -2\log|x|$ ($x \in \mathbb{R}^2$) is sub-diffusive. We also give an explicit formula of the density of the stochastic domination of the Ginibre random point field.

The Ginibre random point field μ is a probability measure on the configuration $\mathbb{S}$ over $\mathbb{R}^2$. It is known that μ is translation and rotation invariant. Moreover, μ is so called a determinantal random point field whose n-correlation function ρ^n is given by

$$\rho^n(x_1,\dots,x_n) = \det[K(x_i,x_j)]_{1\le i,j\le n}, \tag{1}$$

where $K:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{C}$ is the kernel defined by

$$K(x,y) = \frac{1}{\pi}\exp(-\frac{|x|^2}{2} - \frac{|y|^2}{2})\cdot e^{x\bar{y}}. \tag{2}$$

Here we identify $\mathbb{R}^2$ as $\mathbb{C}$ by the obvious correspondence: $\mathbb{R}^2 \ni x = (x_1,x_2) \mapsto x_1 + \sqrt{-1}x_2 \in \mathbb{C}$, and $\bar{y} = y_1 - \sqrt{-1}y_2$ means the complex conjugate under this identification.

Intuitively, μ is a measure interacting $2D$ Coulomb potentials Ψ defined by

$$\Psi(x) = -2\log|x| \quad (x\in\mathbb{R}^2). \tag{3}$$

We remark that the DLR equation for μ does not make sense. However, we will prove that μ is the reversible measure of the unlabeled diffusion $\mathbb{X}_t = \sum_{i\in\mathbb{N}}\delta_{X_t^i}$, where the associated labeled dynamics $\mathbf{X}_t = (X_t^i) \in (\mathbb{R}^2)^{\mathbb{N}}$ is the solution of the infinitely dimensional SDE:

$$dX_t^i = dB_t^i + \lim_{r\to\infty}\sum_{|X_t^i - X_t^j|<r}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \qquad (\mathbf{X}_0 = (x_i)_{i\in\mathbb{N}}). \tag{4}$$

We remark the unlabeled diffusion $\mathbb{X}_t$ is translation and rotation invariant. By (4) one can say μ is a measure with $2D$ Coulomb interaction potential Ψ.

Theorem 1. *There exists a set $\mathbf{S}\subset(\mathbb{R}^2)^{\mathbb{N}}$ such that $\mu(\{\sum_{i\in\mathbb{N}}\delta_{x_i};\mathbf{x}=(x_i)\in\mathbf{S}\}) = 1$ and that (4) has a solution for all initial points $\mathbf{x} = (x_i)_{i\in\mathbb{N}}\in\mathbf{S}$. Moreover, for all initial points $\mathbf{x}\in\mathbf{S}$,*

$$P(\mathbf{X}_t \in \mathbf{S}\cap\mathbf{S}_{\text{single}} \textit{ for all } t) = 1.$$

Here $\mathbf{S}_{\text{single}} = \{\mathbf{s} = (s_i)\in(\mathbb{R}^2)^{\mathbb{N}};\, s_i \neq s_j \text{ if } i\neq j\}$. More precisely, there exist $(\mathbb{R}^2)^{\mathbb{N}}$-valued process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ and Brownian motion $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ such that the pair $(\mathbf{X},\mathbf{B})$ satisfies the SDE (4).

The key point of Theorem 1 is to calculate the log derivative of the one moment measure μ^1 of μ and to introduce a coupling of an infinite system of Dirichlet spaces describing (4). Here μ^1 is the measure on $\mathbb{R}^2\times\mathbb{S}$ such that $\mu^1(A\times B) =$

$\int_A \rho^1(x)\mu_x(B)dx = (1/\pi)\int_A \mu_0(B)$, where μ_0 is the Palm measure of μ conditioned at x.

Theorem 2. *Let α be the self-diffusion matrix of Ginibre interacting Brownian motion* (4)*. Then $\alpha = 0$.*

We remark (4) has only repulsive interaction; there are no center force. If the interaction is of Ruelle class and $d \geq 2$, then α is always non degenerate. This was proved mathematically when the particle have convex hard cores [O. 98, PTRF]. So the result above caused by the strength of the long range part of the interaction potential Ψ.

A key point of the proof is a μ-almost sure equality on functions of the configuration space. We also use the invariant principle and, moreover, the necessary and sufficient condition for the non degeneracy of the limit coefficients under diffusive scaling obtained in [O. 98, IHP].

The second topic is the stochastic domination of the Ginibre random point field μ. By construction μ is a probability measure on $\mathbb{S}$. We construct a measure ν on $\mathbb{R}^2 \times \mathbb{S}$ such that

$$\nu \circ \iota^{-1} = \mu. \tag{5}$$

Here $\iota : \mathbb{R}^2 \times \mathbb{S} \to \mathbb{S}$ defined by $\iota((x, \sum_i \delta_{y^i})) = \delta_x + \sum_i \delta_{y^i}$. The existence of ν is proved by Goldman. He also proved the marginal distribution of the first component of ν is the Gaussian measure on $\mathbb{R}^2$, that is, a probability measure!. We remark the existence of such a probability marginal never happens in the case of translation invariant Ruelle's class Gibbs measure. So, as well as the sub diffusivity of the tagged particles, it reflects the strength at infinity of the 2D Coulomb potential potential. We give here an explicit formula of the density $d\nu/dx \times \mu_0$.