Spectral gap for multi-species exclusion processes

Yukio Nagahata[1]
Department of Mathematical Science
Graduate School of Engineering Science
Osaka University, Toyonaka, 560–8531, JAPAN
Makiko Sasada[2]
Graduate School of Mathematical Science,
The University of Tokyo, Komaba, Tokyo, 153–8914, JAPAN

Let Λ_n be the d-dimensional cube with width $2n+1$, centered at the origin. Fix $r \in \mathbf{N}$ such that $r \geq 2$, we define $\Sigma_n := \{0, 1, 2, \ldots, r\}^{\Lambda_n}$ and $\Sigma_{n,k} := \{\eta \in \Sigma_n; \sum_{x \in \Lambda_n} 1(\eta_x = i) = k_i \text{ for all } 0 \leq i \leq r\}$ for $k = (k_0, k_1, \ldots, k_r)$ with $\sum_{i=0}^r k_i = |\Lambda_n|$. For $\eta \in \Sigma_n$ and $x, y \in \Lambda_n$, we define the configuration $\eta^{x,y} \in \Sigma_n$ by

$$\left(\eta^{x,y}\right)_z = \begin{cases} \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y \\ \eta_z & \text{otherwise,} \end{cases}$$

and the operator $\pi^{x,y}$ by

$$\pi^{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta).$$

Let $\{p_i\}_{0 \leq i \leq r}$ be finite range, translation invariant, irreducible symmetric transition probabilities on $\mathbf{Z}^d$.

Given a function $g : \{0, 1, \ldots, r\} \to \mathbf{R}$ such that $g(i) > 0$, and p_i, we define the infinitesimal generator L_n by

$$(L_n f)(\eta) := \sum_{x,y \in \Lambda_n} p_{\eta_x}(x, y) g(\eta_x) 1(\eta_y = 0) \pi^{x,y} f(\eta)$$

for $f : \Sigma_n \to \mathbf{R}$. The process is regarded as a gas of multi-species particles. The site x is occupied by an i-th particle if $\eta_x = i$ for $1 \leq i \leq r$ and vacant if $\eta_x = 0$. An i-th particle at site x jumps to site y at rate $g(i) p_i(x, y)$ if it is vacant.

Assumption 0.1 There exists n_0 such that for each $n \geq n_0$ and $k = (k_0, k_1, \ldots, k_r)$ with $\sum_{i=1}^r k_i = |\Lambda_n|$ and $k_0 \neq 0$, $\Sigma_{n,k}$ is an ergodic component of L_n.

[1]E-mail address: nagahata@sigmath.es.osaka-u.ac.jp
[2]E-mail address: sasada@ms.u-tokyo.ac.jp

Assumption 0.1 should be rewritten by the condition on the range of $\{p_i\}_{1\le i\le r}$. It seems difficult to give an equivalent condition, hence we give a reasonable sufficient condition:

Lemma 0.2 Suppose that the dimension $d \geq 2$ and $p_i(x) > 0$ for all $\|x\|_1 = 1$ and for all $1 \leq i \leq r$, or $p_i(x) > 0$ for all $\|x\|_1 = 1$ and for all $1 \leq i \leq r$ and there exists $l = (l_1, l_2, \dots, l_r) \in (\mathbf{Z}^d)^r$ such that $\|l_i\|_1 \geq 2$, $p_i(0, l_i) > 0$ for all i, where $\|x\|_1 = \sum_{i=1}^d |x_i|$ for $x = (x_1, x_2, \dots, x_d) \in \mathbf{Z}^d$. Then Assumption 0.1 holds true.

Let $\nu_{n,k}$ be the uniform measure on $\Sigma_{n,k}$. Due to Assumption 0.1, the restriction of L_n on $\Sigma_{n,k}$ which is denoted by $L_{n,k}$ is symmetric with respect to $\nu_{n,k}$. In view of this symmetry the second smallest eigenvalue of $-L_{n,k}$ is given by

$$\lambda = \lambda(n,k) := \inf \left\{ \frac{E_{\nu_{n,k}}[f(-L_n)f]}{E_{\nu_{n,k}}[f^2]} \;\middle|\; E_{\nu_{n,k}}[f] = 0 \right\}.$$

We call λ the spectral gap of $L_{n,k}$.

Theorem 0.3 Suppose that the number of species of particles is at least two, i.e., there are $1 \leq i < j \leq r$ such that $k_i, k_j > 0$. Then we have

$$\lambda(n,k) \asymp \frac{\rho_0}{n^2},$$

where $\rho_0 = \frac{k_0}{|\Lambda_n|}$ and $f \asymp g$ means that there exist positive constants C, C', which do not depend on n nor k, such that $Cf \leq g \leq C'f$.

Remark 0.4 Suppose that the number of species of particles is one. Then this gives a symmetric simple exclusion process. Hence we have

$$\lambda(n,k) \asymp \frac{1}{n^2},$$

i.e., it does not depend on the density of vacant site.