Localization of inhomogeneous coined quantum walks on the line

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As a quantum counterpart of the classical random walk, the quantum walk (QW) has recently attracted much attention for various fields. There are two types of QWs. One is the discrete-time (or coined) walk and the other is the continuous-time one. The discrete-time QW in one dimension (1D) was intensively studied by Ambainis et al. [1]. One of the most striking properties of the 1D QW is the spreading property of the walker. The standard deviation of the position grows linearly in time, quadratically faster than classical random walk. The review and book on QWs are Kempe [2], Kendon [3], Venegas-Andraca [4], Konno [5], for examples. In this talk we focus on discrete-time case. The model mainly considered here [6] is a space-inhomogeneous two-state 1D QW. The two-state corresponds to left and right chiralities. Let $p_n(0)$ denote the probability that the walker returns to the origin at time n. The model is said to exhibit localization if $\limsup_{n\to\infty} p_n(0) > 0$. The homogeneous two-state 1D QW except a trivial case (see [1, 7, 8], for examples) and a class of inhomogeneous two-state 1D QWs [9] do not exhibit localization. The decay order of $p_n(0)$ is closely related to the recurrence. As for the recurrence property of QWs, see Štefaňák et al. [10, 11, 12]. Localization of the homogeneous model was shown for a three-sate 1D QW in [13], a four-state 1D QW in [14], and a multi-state QW on tree in [15]. Mackay et al. [16] and Tregenna et al. [17] found numerically that a homogeneous 2D QW exhibits localization. Inui et al. [18] and Watabe et al. [19] showed the phenomenon. In higher dimensions, a d-dimensional homogeneous tensor-product coin model does not exhibit localization [11]. Oka et al. [20] analyzed localization of a two-state QW on a semi-infinite 1D lattice, which is closely related to the Landau-Zener transition dynamics. Through numerical simulations, Buerschaper and Burnett [21] and Wójcik et al. [22] reported that the dynamics of the two-state 1D QWs exhibits from dynamical localization, spreading more slowly than in the classical case, to linear diffusion like the homogeneous two-state 1D QW as the period of the perturbation is varied. Linden and Sharam [23] investigated a similar inhomogeneous two-state 1D QW where the inhomogeneity is periodic in position.

An interesting question is whether localization emerges even for a simpler inhomogeneous two-state 1D QW compared with the previous models. We give an affirmative answer to the question (Theorem 1 in this abstract). Our very non-trivial result could be useful for quantum information processing by controlling the spreading of the walker. The model has a simple inhomogeneity at the origin depending on a single parameter. Despite of this rather small deviation from the homogeneous Hadamard walk, the presented model differs significantly from the previous ones. In particular, it is shown that it leads to localization. We describe the above mentioned result more precisely. For a given sequence $\{\omega_x : x \in \mathbb{Z}\}$ with $\omega_x \in [0, 2\pi)$, we consider the inhomogeneous 1D QW given by

$$U_x = U_x(\omega_x) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\omega_x} \\ e^{-i\omega_x} & -1 \end{bmatrix},$$

where the subscript x indicates the location. In particular, we concentrate on a simple inhomogeneous model depending only on a one-parameter $\omega \in [0, 2\pi)$ as follows: $U_0 = U_0(\omega)$, and $U_x = U_x(0)$ if $x \neq 0$. So when $\omega \neq 0$, our model is homogeneous except the origin. If $\omega = 0$, then this model becomes homogeneous and is equivalent to the Hadamard walk. For our inhomogeneous two-state 1D QW with the parameter $\omega \in [0, 2\pi)$, we have

THEOREM 1 *([6])*

$$\lim_{n\to\infty} p_{2n}(0) = \left(\frac{2(1-\cos\omega)}{3-2\cos\omega} \right)^2 =: c(\omega).$$

As for the above limit $c(\omega)$, see Fig. 1. If the model is inhomogeneous, i.e., $\omega \in (0, 2\pi)$, then it exhibits localization, i.e., $c(\omega) > 0$. In addition to Theorem 1, we also discuss some results on the related models ([9, 24], for instance).

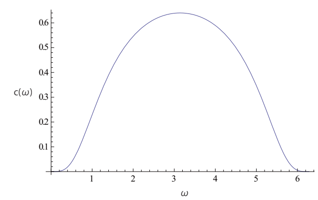

Figure 1: The plot of $c(\omega)$

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